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THE POINCARÉ-LIGHTHILL PERTURBATION TECHNIQUE AND ITS GENERALIZATIONS

by

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ABSTRACT:

The known generalization of the Poincaré-Lighthill perturbation method of strained coordinates are investigated and compared. Some new conditions for its applicability are conjectured and some of its limitations are shown.

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I. INTRODUCTION

What is now referred to as the P-L-K (Poincaré-Lighthill-Kuo) method or the Lighthill method of strained coordinates was first proposed by M. J. Lighthill in 1949 [1] as a perturbation method for obtaining uniformly valid approximate solutions for certain classes of ordinary and partial differential equations. It was then used successfully by many authors (see Lighthill's 1961 lecture [2] for a sample list) although there was a growing feeling of uneasiness on the part of some workers because of cases of partial differential equation problems where the method gave some wrong answers, even though the method looked right (see Van Dyke [3], Levy [4] and Lin [5]).

Because partial differential equations are too hard, mathematical investigations have centered on the ordinary differential equation problems, and in particular, the model equation originally used by Lighthill.

$$(x + \epsilon y)\frac{dy}{dx} + q(x)y = r(x) \qquad 0 \le x \le 1 \tag{1}$$
$$y(1) = b$$

Following a conversation at a cocktail party Wasow [6] proved that Lighthill's method worked for (1) with an added hypothesis. With Wasow's paper the purely mathematical interest died, except for isolated papers which went basically unnoticed. The essential ones are Temple's lecture at the 1958 International Congress [7] where he proposed a modification which amounts to a different motivation of a similar method,

Pritulo's paper [8] which reduces a portion of the problem to an algebraic equation as opposed to a differential equation, and Takahasi's paper [9] which generalized (1) to include higher order non-linearities and proved the convergence of Lighthill's procedure in the manner of Wasow.

Since 1968 however, there has been considerable activity in studying the validity of applying Lighthill's method to the ordinary differential equation case. The activity started with Comstock's paper [10], based on a conversation (at a cocktail party) with Lin, which showed by a series of examples that Lighthill's method was quantatively inaccurate in the more general case,

$$(x^n + \epsilon y)\frac{dy}{dx} + q(x)y = r(x),\ n > 1 \tag{2}$$

and, in addition, the method was also qualitatively inaccurate if Wasow's extra criterion did not hold, although the method appeared to work in all cases. Since that paper, several papers have appeared trying to patch up these difficulties. Usher [11] rediscovered a portion of Pritulo's result [8] and showed where Wasow's criterion seems quite reasonable. Burnside [12], starting from Temple's approach, showed that an initial change of independent variables to reduce the exponent n in (2) to one would solve some of the inaccuracies pointed out by Comstock [10]. Independently Melka [13] arrived at a similar, and slightly more general, result. Also, in an as yet unpublished work [14], Melka suggests a different and potentially interesting approach. And most recently Su and Liu [15] finally attempt to analyze the problem in the framework of a

general asymptotic expansion.

The purpose of this paper is to combine these results to illustrate the different approaches and their inherent problems, correct some errors and to suggest some new results.

II. THE P-L-K METHOD AND ITS RESTRICTIONS

The original model of Lighthill [1] was the problem

$$(x + \epsilon y) \frac{dy}{dx} + q(x)y = r(x) \qquad 0 \le x \le 1 \tag{1}$$

$$y(1) = b > 1 \tag{2}$$

$$q(0) \neq 0 \tag{3}$$

This last condition is one to which we will return later. If one tries to find y as a series in ϵ

$$y \sim y_o(x) + \epsilon y_1(x) + \epsilon^2 \cdot y_2(x) + \dots$$

then the equation for y_o is

$$x \frac{dy_o}{dx} + q(x)y_o = r(x) \tag{4}$$

whose homogeneous part has a regular singular point at the origin. The original equation, as a simple phase plane analysis shows, does not have a singularity at the origin, and thus the perturbation series has a false singularity. (It is the condition (3) which makes this singularity a regular singular point.) The false singularity is disturbing enough, but in addition it is easy to see that the solutions for $y_1(x)$, $y_2(x)$, etc. are even more singular, due to the term of y dy/dx.

Lighthill's idea was to move the singularity back out of the domain of interest by introducing a new, slightly stretched, independent variable z by the implicit equation.

$$x = z + \epsilon x_1(z) + \epsilon^2 x_2(z) + \dots \tag{5}$$

and then look for a solution y in the form

$$y = y_o(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \dots \tag{6}$$

Utilizing (5) and (6) in (4) and expanding the functions r(x) and q(x) about the point z, one generates a single sequence of differential equations. The i^{th} equation is a single linear first order equation in both dy_i/dz and dx_i/dz, in terms of the lower order y_i and x_i. Clearly one must generate a second sequence of equations to determine both the y_i and the x_i. Lighthill's choice is best phrased by Van Dyke [3] who said "higher approximations shall be no more singular than the first." In this instance one groups all the non-homogeneous terms in the i^{th} equation which could contribute to making the solution for $y_i(z)$ more singular than $y_{i-1}(z)$ into a group including all the $x_i(z)$ terms, and set this equal to zero, thus creating a differential equation for $x_i(z)$. This necessitates chosing a boundary condition for $x_i(z)$ as well. One such choice is to make x = 1 correspond to z = 1, i.e.,

$$x_i(1) = 0, \tag{7}$$

although this may not be the best choice.

We reemphasize the choice of grouping of terms to create a

differential equation for x_i. One only need put the singular terms into this equation. One can go so far as to put all the non-homogeneous terms into the equation for $x_i(z)$. Then, with the boundary conditions $y_i(1) = 0$, $i \geq 1$, the linear homogeneous equations for y_i will make all the $y_i \equiv 0$ and equation (6) converges very nicely, consisting only of $y_o(z)$.

Wasow in a paper [6], which was extended and corrected 11 years later by Sibuya and Takahasi [16], showed that the above procedure, choosing $y_i \equiv 0$ $i \geq 1$, led to a convergent series expansion for the $x_i(z)$, provided $q(0) \neq 0$ and also

$$y_o(z)\, q(z) - r(z) \neq 0 \tag{8}$$

$$0 < z \leq 1.$$

This inequality has since been referred to as the Wasow criterion, and did not appear in the original Lighthill work. The paper of Sibuya and Takahasi went a bit further and asked whether this convergent series for $x = z(z, \epsilon)$ could be inverted for z. They prove that in addition if there exists any analytic particular solution $\omega(z)$ to the zero order equation with the property $\omega(1) \neq b$, then the series is invertable. If $\omega(1) < b$, the point $x = 0$ corresponds to a positive value of z and there is no difficulty. If $\omega(1) > b$ then the solution to (1) has a genuine singular point in the domain of interest and the point $x = 0$ corresponds to a negative value of z. In a subsequent paper Takahasi [9] extended these results to the equation

$$\{x + \sum_1^a \epsilon^m P_m(x,u)\} \frac{du}{dx} + q(x)u = r(x) + \sum_1^c \epsilon^m R_m(x,u) \tag{9}$$

where the P_m and the R_m are polynomials. The theorems are essentially the same as the previous paper by this author [16].

Thus the work of Wasow and his successors have led to the following result:

For the equation

$$(x + \epsilon P(u)) \frac{du}{dx} + q(x)u = r(x) \tag{10}$$

$$u(1) = b$$

Lighthill's procedure of expanding both u and x in terms of a slightly strained variable z, given by

$$x = z + \sum_1^\infty \epsilon^n x_n(z), \tag{11}$$

subjecting the $x_i(z)$ to the differential equation naturally generated by using series expansions in (10) and grouping the singular terms, will work provided certain conditions hold. They are

$$\begin{aligned} &\text{i)} \quad q(0) \neq 0 \\ &\text{ii)} \quad u_o(z)\, q(z) - r(z) \neq 0 \text{ in } 0 < z \leq 1 \\ &\text{iii)} \quad b \text{ is large enough} \end{aligned} \tag{12}$$

Three questions immediately arise. Can the procedure be extended to more general equations than (10)? Secondly, must the straining be of the form (11) or would a more general straining work? And lastly, what is the significance of the restrictions (12) listed just above?

Temple's approach to Lighthill's problem [7] certainly suggests that a much more general problem could be attacked. Temple framed the problem in the following way. Rewriting (10) as

$$\frac{du}{dx} = \frac{q(x)u + r(x)}{x + \epsilon u} \tag{10a}$$

one can view this as the result of combining two ordinary differential equations

$$\begin{aligned} \frac{du}{d\xi} &= q(x)u + r(x) \\ \frac{dx}{d\xi} &= x + \epsilon u \end{aligned} \tag{13}$$

Now the differential equation for x is chosen ab initio. Because of the analyticity of the right hand sides of (13) in x, u and ϵ, a series expansion for x and u in powers of ϵ is assured to converge The variable ξ in (12) is related to z in (11) by $\xi = \log z$.

There is a slightly more natural way to get Lighthill's stretched variable z by writing (10a) as

$$\begin{aligned} z\frac{du}{dz} &= q(x)\, u + r(x) \\ z\frac{dx}{dz} &= x + \epsilon\, u \end{aligned} \tag{13a}$$

However these equations have an obvious singularity, where as (13) does not. There is now an elementary jump to the equation

$$[x^k + \epsilon g(x,u,\epsilon)]\frac{du}{dx} + h(x,u,\epsilon) = 0 \tag{14}$$

which can be written

$$\frac{du}{dz} = -h(x,u,\epsilon)$$
$$\frac{dx}{dz} = x^k + \epsilon\, g(x,u,\epsilon) \tag{15}$$

and Bellman [17] makes this jump, saying that Temple's method would work just as well here. As we shall see, this conjecture is false.

Temple's approach actually presents several difficulties, but it also suggests a couple of positive conjectures. The first difficulty is that the differential equation for x is selected with no reference to deleting any singularities for the expansion of u. This removes the original motivation for the problem. In contrast to the series used by Wasow, both the u series and the x series generated by this split will, in general, be infinite series. The flexibility to make the u expansion a finite number of terms is gone. A second difficulty is that the Wasow restrictic s (12) are all on the equation for u, which now does not appear to have any problem. The biggest difficulty, however, is that the singularity involved in the equation is hidden. The equations (13) are actually analytic in u, x and ϵ, provided z is finite. But the zero order solution for x is

$$x_o = e^z \tag{16}$$

and the compact domain $0 \le x \le 1$ is mapped into the semi-infinite domain $-\infty \le z \le 0$, and unfortunately it is the point at infinity which is the point of primary interest for most of the problems. We will return to this point later.

Another splitting of the equations, starting from the Lighthill approach,

was devised by Pritulo [8] and later again by Usher [11]. A simple observation reduces the problem of finding $x_i(z)$ to an algebraic problem rather than a differential equation problem. This observation is the following theorem [8].

Theorem: If an ordinary perturbation expansion of u alone has a (non-uniformly convergent) expansion

$$u \sim \sum_{n=0}^{\infty} \epsilon^n u_n(x), \tag{17}$$

then the Lighthill expansion for u is given by the formula

$$u \sim \sum_{n=0}^{\infty} \epsilon^n U_n(z) \tag{18}$$

where

$$U_n(z) = u_n(z) + \sum_{n=0}^{n-1} c(z)_{n-m-1} \frac{d}{dz} u_m(z) \tag{19}$$

with $c_o(z) = x_1(z)$

$$c_p(z) = \frac{1}{px_1(z)} \sum_{k=1}^{p} (2k-p)\, x_{k+1}(z)\, c_{p-k}(z) \tag{20}$$

where $$x = z + \sum_{1}^{\infty} \epsilon^n x_i(z) \tag{21}$$

The power of this theorem is that these equations (19-21) hold for any choices of the $x_i(z)$. Thus we can make certain choices. By hypothesis the $u_n(x)$ form an increasingly singular set, so the $U_n(z)$ do also. However, in (19), the $x_i(z)$ can be chosen successively so that they subtract off the increasingly singular part, so that the $U_n(z)$ can be made

uniformly convergent!

Prit ulo extended his formula to higher order differential equations, but he made no applications of his results. He was also apparently not aware of Wasow's paper.

Several years later Usher made a similar observation [11]. Usher started from the equation

$$\frac{dy}{dx} = f(x, y, \epsilon) \tag{22}$$

where y may be a vector and assumed that (22) was analytic in ϵ so that it may be expanded in the form

$$\frac{dy}{dx} = f_o(x, y) + \epsilon\, f_1(x_1 y) + \ldots \tag{22a}$$

Looking at the model equation (1) in this way

$$\frac{dy}{dx} = \frac{q(x)u + r(x)}{x + \epsilon\, u} \qquad 0 \leq x \leq 1$$

$$= \frac{1}{x}\,[q(x)u + r(x)] + \frac{\epsilon u}{x^2}\,[q(x)u + r(x)] + \ldots$$

we see that the expansion is invalid in the given domain. However, Usher only wanted the first order terms in u and x, and the lack of analyticity in ϵ does not affect these calculations. Usher then looked at the zero and first order terms, in light of Wasow's results. We write them out, in Pritulo's notation

$$\begin{aligned} U_o(z) &= u_o(z) \\ U_1(z) &= u_1(z) + x_1(z)\,\frac{du_o}{dz} \end{aligned} \tag{23}$$

But since, for the model equation,

$$\frac{du}{dz} = \frac{1}{x}\{q(z)u_o + r(z)\}$$

we have

$$U_1(z) = u_1(z) + \frac{x_1}{z}[q(z)u_o + r(z)]. \tag{24}$$

Separating $u_1(z)$ into its very singular and mildly singular parts $u_1(z) = u_{1m}(z) + u_{1s}(z)$ we see that there is a solution of (24) for $x_1(z)$ which leaves $U_1(z)$ only mildly singular provided that Wasow's criterion holds. Thus it was Usher who realized that the restriction ii) in (12) (Wasow's criterion) is necessary criterion for a finite $x_1(z)$ throughout the range $0 < z \leq 1$. And equation (19) shows that $z^{-1}[q(z)u_o + r(z)]$ is the coefficient of each succeeding $x_i(z)$, so this restriction applied to each succeeding term.

Does the singularity in the expression for x_1 invalidate the use of the method? Apparently not, to first order, as the following example shows. We consider a problem which is explicitly solvable, namely

$$\begin{aligned} &(x + \epsilon u)\frac{du}{dx} = -u + 2b\,x \qquad\qquad 0 \leq x \leq 1 \\ &u(1) = B = b(1 + \eta) \end{aligned} \tag{25}$$

By simple integration, since the equation is exact, we have

$$u = \epsilon^{-1}\{-x + \sqrt{x^2 + 2\epsilon(\bar{c} + bx^2)}\}$$

with $$\bar{c} = b\eta + \epsilon\frac{b^2}{2}(1 + \eta)^2 \tag{26}$$

Solving this problem using Usher's approach, we write

$$u(x) = u_o(x) + \epsilon u_1(x) + \ldots$$

$$u_o(1) = B \tag{27}$$

$$u_i(1) = 0 \qquad i \geq 1$$

this gives

$$u_o(x) = bx + b\eta x^{-1} \tag{28}$$

$$u_1(x) = \frac{b^2}{2}\{(1+\eta^2)x^{-1} - \eta^2 x^{-3} - x\} \tag{29}$$

and the Wasow criterion term is

$$-u_o + 2bx = b(x - \frac{\eta}{x}) \tag{30}$$

We recognize that $u_1(x)$ is more singular than $u_o(x)$, and also that (30) vanishes in the domain $0 < x < 1$ if $0 < \eta < 1$. Using Usher's approach we rewrite

$$U_o(z) = b(z + \frac{\eta}{z}) \tag{28a}$$

$$u_1(z) = \frac{b^2}{2}\{(1+\eta^2)z^{-1} - z - \eta^2 z^{-3}\} \tag{29a}$$

$$= U_1(z) - x_1(z)b\{1 - \frac{\eta}{z^2}\}$$

There is some flexibility in the grouping of (29a) to create $U_1(z)$. We choose to make $U_1(1) = 0$ and make the choice

$$U_1(z) = \frac{b^2}{2}(1+\eta^2)(z^{-1} - 1) \tag{31}$$

so that

$$x_1(z) = \frac{b}{2}\{\eta^2(z^{-3} - 1) + (z-1)\}\{1 - \frac{\eta}{z^2}\}^{-1} \tag{32}$$

and $x_1(z)$ is indeed singular at $z = \sqrt{\eta}$.

However, what information do we wish to know? We claim that at this stage the only important questions are a) does the graph of U_o approximate that of the true u, and b) what is the value of u(0) predicted by our approximation. Ignoring the singularity in $x_1(z)$ we can graph the sum $U_o(z) + \epsilon U_1(z)$, and since u_o is no more singular than $U_o(z)$ it will not alter the graph. Now $U_o(z)$ is strictly positive and has a minimum at $z = \sqrt{\eta}$ where its value is $U_o(\sqrt{\eta}) = 2b\sqrt{\eta}$. But an investigation of the exact solution (26) shows that u(x) is strictly positive, and has a zero slope at $x_o = \sqrt{\eta}(1 + \epsilon b)^{-1}$, where its value is $u(x_o) = 2b\sqrt{\eta}\ (1 + \epsilon b)^{-1}$. And if we assume that x = 0 still corresponds to z very small then we can solve the equation,

$$0 = x = z + \epsilon x_1(z) = z + \frac{b}{2}\{\eta^2(z^{-3}-1) + (z-1)\}\{1 - \frac{\eta}{z^2}\}^{-1} \tag{33}$$

for z, at least approximately. We obtain

$$z(0) \approx \sqrt{\eta b \frac{\epsilon}{2}}$$

so that

$$u_o(x = 0) \approx \frac{b\eta}{z} = \sqrt{\frac{2b\eta}{\epsilon}} \tag{34}$$

as compared with the exact value from (26)

$$u(0) = \sqrt{\frac{2\bar{c}}{\epsilon}},\ \bar{c} = b\eta + \frac{\epsilon b^2}{2}\ (1 + \eta^2) \tag{35}$$

Thus these features of the approximation fit quite well.

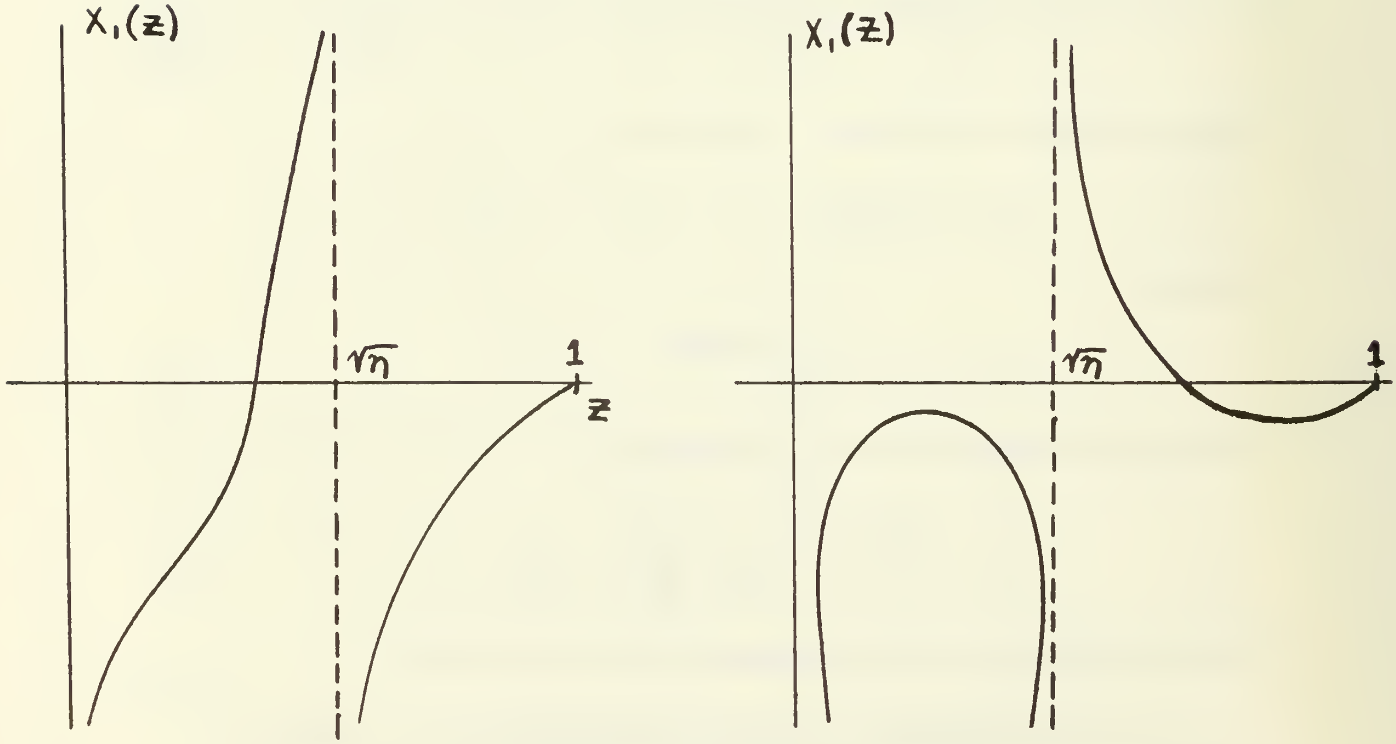
Is our assumption that z is small corresponds to x = 0 valid, in view of the singularity in $x_1(z)$? A study of $x_1(z)$, equation (32), shows

that this is valid. An elementary analysis of the numerator of (32) shows that $x_1(z)$ has two positive zeros, one at $z = 1$ by construction, and one at some smaller value of z, depending upon η. The details of the behavior depend upon whether this zero is greater than or less than the singular point $z = \sqrt{\eta}$. We sketch the graph for the two cases. The critical value where the zero and the singular point coincide is given by looking for the switch or sign of $\frac{dx}{dz}$ near $\sqrt{\eta}$ which occurs at

$$1 + \eta^2 = 2\sqrt{\eta} \tag{36}$$

or

$$\eta = .296 \tag{36a}$$



The fact that this example gives a number of right answers despite the extreme distortion of the "stretched" coordinate encourages us to ask whether the distortion can be corrected. To see that it can, we return to a comment made earlier concerning the flexibility on choosing $x_1(z)$ and $U_1(z)$. In particular, there is no necessity for making x and z coincide at x = 1. (Doing so vastly simplifies the assigning of the condition u(x = 1) = B, of course.) In particular one can make x and z coincide at the singular point $x = \sqrt{\eta}$, as we show for this example. Going back to (29a) we write

$$
\begin{aligned}
u_1(z) &= \frac{b^2}{2}\{(1+\eta^2)z^{-1} + 2z - 3z - \eta^2 z^{-3} + 4\sqrt{\eta} - 4\sqrt{\eta}\} \\
&= \frac{b^2}{2}\{(1+\eta^2)z^{-1} + 2z - 4\sqrt{\eta}\} - \frac{b^2}{2}\{\eta^2 z^{-3} + 3z - 4\sqrt{\eta}\}
\end{aligned}
\tag{29b}
$$

Now we have

$$
x_1(z) = \frac{b}{2}\{\eta^2 z^{-3} + 3z - 4\sqrt{\eta}\}\{1 - \eta z^{-2}\} \tag{37}
$$

which has the limit

$$
x_1(\sqrt{\eta}) = 0 \tag{38}
$$

Obviously all higher order $x_i(z)$ can be treated in a similar fashion. And we observe that we have made the zero of x_1 a simple zero at $z = \sqrt{\eta}$, so x_1 is negative to the left of $x = \sqrt{\eta}$ and positive to the right, as it should be.

To obtain the correct boundary conditions for u in the series form (27) we can not take the $u_i(z) = 0$ at x = 1, as we did. This means

that we must go back one step further and take the indefinite integral for (29b), replacing $(1 + \eta^2)$ by c_1. Then we have

$$U_1(z) = \frac{b^2}{2} \{c_1 z^{-1} + 2z - 4\sqrt{\eta}\} \tag{39}$$

To determine c_1 we must examine the equation

$$1 = z + \epsilon x_1(z) \tag{40}$$

and solve for z. We get

$$z(1) = 1 + \epsilon d_1 + 0(\epsilon^2)$$

Then we look at

$$\begin{aligned} U(x = 1) &= U_o\big|_{x=1} + \epsilon u_1\big|_{x=1} + 0(\epsilon^2) = B \\ &= b[1 + \epsilon d_1 + \eta(1 + \epsilon d_1)^{-1}] \\ &\quad + \epsilon \frac{b^2}{2} [c_1(1 + \epsilon d_1)^{-1} + 2(1 + \epsilon d_1) - 4\sqrt{\eta}] \end{aligned} \tag{41}$$

Equating powers of ϵ we get the condition for c_1 to be

$$c_1 = 4\sqrt{\eta} - 2 - \frac{2}{b} d_1 (1-\eta) \tag{42}$$

Higher order terms may be computed in an analogous fashion.

We now conjecture the theorem:

Theorem: In the case that the zero order term in the Lighthill expansion for (1) has a single zero slope at some x_o in the interior $0 < x_o < 1$, then the series expansions may still be made to converge by making x = z at

x_o. The theorem would be much more difficult to prove than the Wasow result, because by necessity both the x series and the u series are infinite series.

The fact that it might be advantageous not to make $z = x$ at $x = 1$ was observed by Melka [13] in his thesis, but the reasons for making any particular choice were not discussed in any detail.

We now turn to the question of the restriction $q(0) \neq 0$ (Eq (3)) in Lighthill's original problem. Wasow also makes the restriction in his theorem (12i). Lighthill's choice is based upon the character of the singular point in the zeroth order equation near $x = 0$, i.e.,

$$x \frac{du}{dx} + q(0)u = r(0)$$

has a solution which behaves like $u \approx A_o x^{-q(0)}$ near the origin, and the case $q(0) > 0$ is of different character than $q(0) < 0$. However, several recent papers have, without saying so explicitly, suggested that the restriction is more fundamental than Lighthill suggested.

Comstock's [10] examples were of the form

$$[x^n + \epsilon u] \frac{du}{dx} + nx^{n-1} u = mx^{m-1} \tag{43}$$

$$u(1) = b > 1$$

For $n > 1$ the zeroth order equation has the same character as Lighthill's case for $q > 0$. However, even in the cases where Wasow's criterion [12ii] held and the P-L-K expansion of (43) had the correct qualitative behavior, the P-L-K solution to (43) give

$$u(0) \approx \sqrt{n} \sqrt{\frac{2(b-1)}{\epsilon}}$$

as opposed to the exact expansion

$$u(0) = \sqrt{\frac{2(b-1)}{\epsilon}}$$

The error is a multiplicative factor of $\sqrt{n}$.

Carrier has given a famous example [18] to show that the zero order equation need not be linear in x. He studied several equations of the form

$$(x^2 + \epsilon u)\frac{du}{dx} + \frac{2}{\alpha}u = r(x) \tag{44}$$

For the case r = 0 this equation is exactly integrable in terms of Bessel functions and one obtains the implicit solution

$$x = -(\epsilon u)^{\frac{1}{2}} \frac{\{J_1(\alpha\sqrt{\epsilon u}) + A\,Y_1(\alpha\sqrt{\epsilon u})\}}{\{J_o(\alpha\sqrt{\epsilon u}) + A\,Y_o(\alpha\sqrt{\epsilon u}\}} \tag{45}$$

with the asymptotic approximation

$$u(0) \approx [\epsilon \ln \frac{1}{\epsilon}]^{-1} \tag{46}$$

The P-L-K method works quite satisfactorily to give this result also.

Burnside [12] in trying to correct the discrepancy of Comstock's examples for $n \neq 1$, took the approach that the essential feature is to linearize the x dependence in the coefficient of $\frac{du}{dx}$. He always makes the change of variables

$$z = x^n \tag{47}$$

so that (43) becomes

$$(z + \epsilon u)\frac{du}{dz} + u = \frac{m}{n} z^{\frac{m}{n} - 1} \tag{48}$$

and then he expands z in a Lighthill expansion. In this way he concludes that the $\sqrt{n}$ discrepancy comes from the neglect of n terms of the same order due to an expansion of a term raised to the n^{th} power. We note that (48) has $q(0) \neq 0$ whereas (43) does not

Melka [13], independently of Burnside,tried to eliminate the discrepancy in Comstock's examples by another change of independent variables. He made the choice for a Lighthill expansion of

$$x^k = z^k + \sum_1^\infty \epsilon^n x_n(z) \tag{49}$$

where k is to be chosen in some fashion which is never specified except by example. For the Comstock examples he noted that a boundary layer approximation near $x = 0$ suggested that $k = n$ is the "logical" choice. This choice will also eliminate the discrepancy.

Melka also considered a generalization of the Carrier problem

$$(x^n + \epsilon u)\frac{du}{dx} + \alpha x^p u = 0 \tag{50}$$

in the case $n = 2(p + 1)$. This equation is a Ricatti equation for x^{p+1} and then Melka claims that an obvious choice in (49) is $k = p + 1$. (Note for Carrier's example this gives just the ordinary Lighthill expansion.) His analysis again is based on knowing the exact solution.

We suggest that there is another basis for a choice of change of coordinates, a choice that will make the Lighthill method valid. Our choice is the following: given the equation

$$(x^n + \epsilon u)\frac{du}{dx} + q(x)u = r(x) \tag{51}$$

with $q(x) \sim ax^p$ as $x \to 0$, <u>first</u> make the change of variables

$$y = x^{p+1}$$

<u>so as to make</u> $q(y) \to a$, <u>a constant</u> $\neq 0$ as $y \to 0$. In this new coordinate system Lighthill's restriction (3) is met. In all the known exactly integrable cases the author can find, the subsequent Lighthill approximation is valid. In other cases the Lighthill procedure appears to work, but we are unable to check our answers against a known solution. It is the property $q(0) \neq 0$ that we feel is the important criterion in the choice of variables, not the resultant linearity or non-linearity of the coefficients.

III. GENERALIZATIONS AND CONCLUSIONS

The original interest in the PLK method was for applications to partial differential equations. There are essentially no theoretical results concerning the application to partial differential equations. The primary results are examples and counterexamples. Lin [5] and Van Dyke [3] have concluded that this method works quite well in hyperbolic equations, where the characteristics become the "strained" coordinates. For many years it was recognized that the application of this method to elliptic equations usually led to erroneous results [2], [3]. However, in recent years two papers have appeared deriving successful approximations for elliptic problems where previous workers failed [19], [20]. We will sketch the results obtained. The problem is that of flow around a thin airfoil, which can be described by finding a harmonic function

$\varphi(x,y)$ satisfying the following conditions:

$$\varphi \to Ux + O(1) \qquad \text{as } x^2 + y^2 \to \infty$$

$$\frac{\varphi_y}{\varphi_x} = \frac{df}{dx} \qquad \text{on the surface } y = f(x)$$

where $f(x) = \epsilon h(x)$. In particular one choice for $f(x)$ is $f(x) = \pm\epsilon(1 - x^2)^{\frac{1}{2}}$, a very narrow ellipse. The structure of the mathematical problem is significantly different from the ordinary differential equation problems, but the same heuristic is still present. For $\epsilon \neq 0$ the problem has no singularity in the domain of definition (the singularity is the "source" and "sink" inside the thin wing which "create" the wing). In the limit $\epsilon = 0$ the singularity is at the edge of the domain. An attempted perturbation expansion for the solution in powers of ϵ times a function of the complex variable z exhibits the familiar problem of increasingly singular terms.

What is the straining of coordinates which is appropriate to this problem? After several erroneous attempts by several people, Hoogstraten [19] recognized that a key was the preservation of analyticity. He asks that

$$z = \eta + \sum_{1}^{\infty} \epsilon^n z_n(\eta)$$

map the body exactly onto the $-1 \leq \Im m\, \eta \leq 1$. This forms his set of "boundary" conditions for the z_n, in a fashion very analogous to setting $z = 1 \Longleftrightarrow x = 1$ in the ordinary differential equation case.

Martin [20] applies a somewhat different criterion, again based on

analyticity. He considers the same problem but looks at the expansion

$$z = \eta + \sum_{n=1}^{\infty} \epsilon_n z_n(\overset{\eta}{\cancel{\xi}})$$

as a case where one can use the Lagrange inversion formula. He then adopts Pritulo's formula for the expansion of the dependent variable. Thus he is not thinking in terms of a choice of boundary conditions for the unknown stretching, but of an algebraic criterion for this choice. He makes the following choice. Looking at the sequence of equations for the dependent variables choose $\varphi_n(\eta) = a_n \varphi_1(\eta)$ where φ_1 is the lowest order solution, and the a_n are (unknown) constants. It is easy to show, for linear equations, that this can always be done. The result is that the solution for φ sums exactly to

$$\varphi(\eta) = g(\epsilon)\, \varphi_1(\eta)$$

where $g(\epsilon) = \Sigma a_n \epsilon^n$. This gives a system of algebraic equations for the $z_n(\eta)$, as in Usher's [11] paper. Martin gives no criterion for the choice of the constants a_n, and he gives examples where some choices give the right answers and other equally reasonable choices give non-uniformly valid answers. It is not at all apparent how to chose correctly since his choices are based on algebraic convenience and knowing the answer rather than some physically justifiable a priori criterion. The examples give no firm clue as to a choice.

We may then conclude the following. For the ordinary differential equation problem, the hypotheses of the original Wasow proof are not

only sufficient but probably also necessary, with one exception. It appears that we may be able to correct for one zero in the function $y_o(z)\, q(z) - r(z)$ (see (8)) by a change in the stretching. Thus Usher's "necessary condition" are not as necessary as he thought [13]. For the generalized equation (2) it appears that the Lighthill procedure will work with a somewhat different change of variables, namely (49) where k is chosen to ensure that the criterion $q(z) \neq 0$ when $z \to 0$. For the partial differtial cases, in particular elliptic problems, it appears that the essential feature, for linear problems, is to make an analytic change of variables, although there is not yet any analogue of the Wasow paper.

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13. ABSTRACT

The known generalization of the Poincaré-Lighthill perturbation method of strained coordinates are investigated and compared. Some new conditions for its applicability are conjectured and some of its limitations are shown.

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